Tensor product algebras in type A are Koszul

Ben Webster[1]
Department of Mathematics, Northeastern University, Boston, MA

Abstract. In this note, we prove the Koszulity of the tensor product algebra $T^{\underline{\lambda}}$ defined in [Weba, §2] for $\mathfrak{g} = \mathfrak{sl}_n$ and $\underline{\lambda}$ a list of fundamental weights. This is achieved by constructing a graded equivalence of categories between $T^{\underline{\lambda}}$-modules and a sum of blocks of category $\mathcal{O}$ in type A.

Introduction

In [Weba, §2], the author defined graded algebras $T^{\underline{\lambda}}$ associated to a list of highest weights for a Kac-Moody algebra. The representation theory of this algebra is a "categorification" of the tensor product $V_{\underline{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell}$. In [Weba, §4], it is proven that when the base field of this algebra is $\mathbb{C}$ and each λ_i is fundamental, this category of representations is equivalent to a sum of blocks of parabolic category $\mathcal{O}$. Unfortunately, this equivalence is only proven without reference to the grading on $T^{\underline{\lambda}}$-mod and the Koszul graded lift of category $\mathcal{O}$. We would like to know that this equivalence induces an graded equivalence of categories, which is not at all obvious.

We prove this by constructing the Koszul dual version of this equivalence. That is, we prove $T^{\underline{\lambda}}$ is isomorphic to the Ext-algebra of a semi-simple object in the derived category of an already known Koszul algebra: a different sum of blocks of parabolic category $\mathcal{O}$ in type A. This isomorphism is graded by construction.

Throughout, we will call a finite dimensional algebra **Koszul** if it has a graded Morita equivalence with a positively graded Koszul algebra as defined in [BGS96]. That is, A is Koszul if each gradable simple module S_i over A has a choice of graded lift such that $\mathrm{Ext}^j(S_i, S_k)$ is pure of degree j for all simples.

Theorem 1. *There is an isomorphism of graded algebras $T^{\underline{\lambda}} \cong \mathrm{Ext}(S, S)$ where S is a fixed semi-simple generator in the derived category of a particular block of parabolic category $\mathcal{O}$. In particular, the algebra $T^{\underline{\lambda}}$ is Koszul with its originally defined grading.*

We should note that this is actually a new proof of the parabolic-singular Koszul duality in type A (though this requires a great deal of machinery). By Theorem 1, the graded algebra $T^{\underline{\lambda}}_\nu$ is the homological dual of a block of parabolic category $\mathcal{O}$ in type A. On the other hand, [Weba, 4.7] gives an isomorphism with the endomorphisms of a projective generator in a different block of parabolic category $\mathcal{O}$.

Our proof uses results of [Webb] to define a cover $\tilde{T}^{\underline{\lambda}}_\nu$ of $T^{\underline{\lambda}}_\nu$ as a convolution algebra in Borel-Moore homology. This realizes the algebra $\tilde{T}^{\underline{\lambda}}_\nu$ as the Ext algebra of a collection of semi-simple objects in the category of D-modules on the moduli space of representations of a quiver. We can then utilize quantum Hamiltonian reduction

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and results of Losev [Los12] to define a functor from the appropriate category of D-modules to the representations of a parabolic W-algebra. This allows us to construct a sum of shifts of semi-simple modules Y in the derived category of $\mathcal{W}_e^r$-mod whose Ext-algebra is $T^{\underline{\lambda}}$. Previous results of the author [Web11] define an equivalence of categories from the category generated by Y to a sum of blocks of parabolic category $\mathcal{O}$.

This result can also be deduced from results of Hu and Mathas. They define an algebra they call **the quiver Schur algebra** which they prove to be Koszul in [HM, Th. C] (unfortunately, this has produced a terminology clash with [SW], since Morita equivalent but non-isomorphic algebras are given the same name there). These algebras are graded Morita equivalent to the tensor product algebras discussed here, a fact which will be proved in the next revision of [Weba].

The geometry of quivers

Let Γ be the linear quiver with $n-1$ nodes, with nodes numbered $1, 2, \dots, n-1$. We orient the edges in increasing order. For some dominant weight λ, let Γ_λ be the Crawley-Boevey quiver of Γ. This is the union of the quiver Γ with a new node which we label 0 along with $\lambda^i := \alpha_i^\vee(\lambda)$ new edges between 0 and i; if $\underline{\boldsymbol{\lambda}}$ is a sequence of fundamental weights summing to λ, we can order the edges so that if $\lambda_j = \omega_{r_j}$, the jth edge connects with the node r_j. We orient all new edges in toward 0. Let Ω denote the oriented edge set of this quiver.

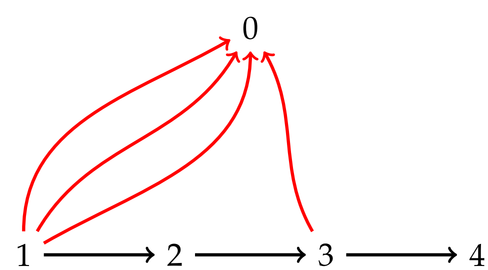

Figure 1. The Crawley-Boevey quiver of $3\omega_1 + \omega_3$ for $\mathfrak{sl}_5$.

For a weight $\nu = \sum d_i\alpha_i$ of $\mathfrak{sl}_n$, we let $V_i = \mathbb{C}^{d_i}$, with $V_0 \cong \mathbb{C}$ always; we also denote $V = \oplus_i V_i$. We let

$$E_\nu = \bigoplus_{e\in\Omega} \operatorname{Hom}(V_{t(e)}, V_{h(e)}).$$

This vector space has a natural action of $G_\nu = \prod_{i\in\Gamma} GL(V_i)$ by pre- and post-composition. The vector $\mathbf{d} = (d_i)_{i\in\Gamma}$ is called the **dimension vector**, and we will freely identify $\mathbb{Z}^\Gamma$ with the root lattice $X(\Gamma)$ by sending $\mathbf{d} \mapsto \nu = \sum d_i\alpha_i$.

The sequence $\underline{\boldsymbol{\lambda}}$ encodes a commuting $\mathbb{C}^*$-action on E_ν; the group $\mathbb{C}^*$ acts trivially on $\operatorname{Hom}(V_i, V_j)$ for $i, j \neq 0$, and with weight i on the ith new edge. Call the image of this action $\mathbb{T}$. We can also think of this as a weighting in the sense of [Webb].

A sequence $(\mathbf{i}, \kappa)$ can be converted into a loading in the sense of [Webb] for the graph Γ_λ. The red lines are placed at $1, 2, \dots, \ell$, and the black lines numbered from

$\kappa(j)$ to $\kappa(j+1)-1$ are spaced evenly between j and $j+1$ (the spacing doesn't matter, since the weight of the edges between i and j have weight 0).

Consider the variety $X_{\mathbf{i},\kappa}$ of quiver representations of Γ_λ, with a flag F^j_k on each V_j vertex such that

- $\dim F^j_k/F^j_{k-1} = \delta_{j,i_k}$,
- the sum $\oplus_{j\in\Gamma} F^j_k$ is a quiver representation of Γ.
- the map along the mth new edge kills F^j_k for $k \leq \kappa(m)$.

These varieties are special cases of the varieties attached to loadings defined in [Webb]; they also appeared in the work of Li [Li] and are denoted $\tilde{\mathbf{E}}_{\underline{\mathbf{a}}}$ in that paper.

This has an obvious map $p_{\mathbf{i},\kappa}\colon X_{\mathbf{i},\kappa} \to E_\nu$ given by forgetting the flag. We let

$$Y_{\mathbf{i},\kappa} = (p_{\mathbf{i},\kappa})_* \mathcal{O}_{X_{\mathbf{i},\kappa}}[\mathbf{u}(\mathbf{i},\kappa)] \qquad \mathbf{u}(\mathbf{i},\kappa) = \dim X_{\mathbf{i},\kappa}$$

be the D-module arising from pushing forward the structure sheaf of $X_{\mathbf{i},\kappa}$, with a shift to preserve self-duality. The Riemann-Hilbert correspondence sends these D-modules to the constructible complexes of sheaves denoted $L_{\underline{\mathbf{a}}}$ in [Li].

Theorem 2 ([Webb, 4.9]). *There is an fully faithful functor from the category $D(\tilde{T}^{\underline{\lambda}}_{\lambda-\nu}$-dg-mod$)$ to the category of G_ν-equivariant D-modules on E_ν, sending $P_{\mathbf{i},\kappa}$ to $Y_{\mathbf{i},\kappa}$.*

This gives us a geometric interpretation of the category $D(\tilde{T}^{\underline{\lambda}}_{\lambda-\nu}$-dg-mod$)$ as complexes of equivariant D-modules on E_ν. We wish to study these modules microlocally. As usual, we can identify T^*E_ν with the analogous representation space of the double of the quiver Γ_λ. The space $X_{\mathbf{i},\kappa}$ has a doubled analogue $Z_{\mathbf{i},\kappa}$, with a map $Z_{\mathbf{i},\kappa} \to T^*E_\nu$. This is the space of representations of the preprojective algebra of Γ_λ, with a flag F^j_k on V_j for each vertex such that

- $\dim F^j_k/F^j_{k-1} = \delta_{j,i_k}$,
- the sum $\bigoplus_{j\in\Gamma} F^j_k$ is a representation of the doubled quiver of of Γ.
- the map to 0 along the mth new edge kills F^j_k for $k \leq \kappa(m)$, and image of the map from 0 along the mth new edge lands in F^j_k for $k \leq \kappa(m)$.

Following Li [Li, §8], we let $\Lambda_{V;\underline{\lambda}}$ denote the union of the images of these maps.

Proposition 3 ([Li, 8.2.1(2)]). *The characteristic variety of $Y_{\mathbf{i},\kappa}$ lies in $Z_{\mathbf{i},\kappa}$.*

Quantum Hamiltonian reduction and the Maffei isomorphism

Let D_ν denote the ring of differential operators on E_ν; this is equipped with a natural action of G_ν. In fact, this is an inner action via a noncommutative moment map $\mu\colon U(\mathfrak{g}_\nu) \to D_\nu$ sending an element of the Lie algebra to the Lie derivative of its infinitesimal action. For each character $\chi\colon \mathfrak{g} \to \mathbb{C}$, we have another such moment map given by $\mu + \chi$ (see [BPW, §3.4] for a more detailed discussion of this set up).

For each χ, we have an algebra

$$A^\chi_\nu = (D_\nu/D_\nu \cdot (\mu+\chi)(\mathfrak{g}_\nu))^{G_\nu},$$

called a non-commutative Hamiltonian reduction equipped with a reduction functor $h_\chi \colon D_\nu\text{-mod} \to A_\nu^\chi\text{-mod}$ defined by

$$h_\chi(M) = \{m \in M \mid (\mu + \chi)(X) \cdot m = 0 \text{ for all } X \in \mathfrak{g}_\nu\}.$$

Let $\chi(\mathfrak{g}_\nu)$ denote the character space of this Lie algebra.

Proposition 4 ([Los12, 5.3.3])**.** *There is a parabolic P_r and nilpotent orbit O_e of $\mathfrak{sl}_N$ for some N and a canonical isomorphism of affine spaces $\gamma\colon \chi(\mathfrak{g}_\nu) \cong \mathfrak{h}^{W_r}$ such that the ring A_ν^χ is a central quotient of the parabolic W-algebra $\tilde{\mathcal{W}}_e^r$ for the central character corresponding to $\gamma(\chi)$ under the Harish-Chandra homomorphism.*

There's a natural map $Z(\tilde{\mathcal{W}}_e) \cong Z(U(\mathfrak{sl}_N)) \to Z(\tilde{\mathcal{W}}_e^r)$; the principal character χ_0 of $Z(U(\mathfrak{sl}_N))$, defined by the action on the trivial representation, factors through $Z(\tilde{\mathcal{W}}_e^r)$. For the remainder of the paper, we let $\eta = \gamma^{-1}(\chi_0)$ denote the inverse image of the principal character of $\tilde{\mathcal{W}}_e^r$. We let $\mathcal{W}_e^r$ denote the quotient of the parabolic W-algebra by the kernel of the principal character. Thus, the theorem above gives an isomorphism $A_\nu^\eta \cong \mathcal{W}_e^r$.

One can regard Theorem 4 as a quantization of the isomorphism between quiver varieties and nilpotent slices given by Maffei [Maf05]. This will not be of significance for our purposes, but let us briefly describe these nilpotents:

- O_e is the nilpotent with λ^i Jordan blocks of length i.
- One takes the partition diagram with λ^i columns of height i, and fills this with the superstandard tableau (each box is filled with the number of its row), and then changes d_i boxes with entry i to having entry $i+1$ for each i (which boxes don't matter for us). If r_j is the number of j's in the resulting tableau, then P_r is the parabolic that preserves a flag with $\dim F_k/F_{k-1} = r_k$.

Under the Maffei isomorphism, the $\mathbb{T}$-action on E_ν is intertwined with the action on the Slodowy slice induced a cocharacter of $GL(n, \mathbb{C})$; we define this cocharacter by choosing a decomposition into Jordan blocks for the action of e (which is not canonical, but unique up to automorphism), and let $\mathbb{C}^*$ act with weight 1 on one Jordan block of length r_1, with weight 2 on a Jordan block of length r_2, etc. That is, we choose one from the unique conjugacy class of cocharacters commuting with e such that the weight spaces of the action are indecomposable as $\mathbb{C}[e]$ modules, and lengths of the blocks read off the indices of the fundamental weights $\lambda_1, \lambda_2, \dots, \lambda_\ell$. We denote the image of the Lie algebra of $\mathbb{C}^*$ in differential operators by $\mathfrak{t} \cong \mathbb{C}$. We let $\mathfrak{q}$ denote the parabolic given by the sum of non-negative weight spaces of this cocharacter.

This allows us to define a category $\mathcal{O}'_{\mathfrak{p}_r}(\chi_0, \mathcal{W}_e^r, \mathfrak{q})$ as in [Web11, Def. 6], which is a version of category $\mathcal{O}$ for the W-algebra (but not the same as that of Brundan, Goodwin and Kleshchev [BGK08], as explained in [Web11]). Let $\mathcal{Q}$ denote the category of A_ν^η-modules that arise by transferring these by the isomorphism $A_\nu^\eta \cong \mathcal{W}_e^r$.

Proposition 5 ([Web11, Th. 8])**.** *The category $\mathcal{Q}$ is equivalent to a singular block of $\mathfrak{p}_r$-parabolic category $\mathcal{O}$ in the sense of BGG. In particular, this category is Koszul by* [BGS96].

The proof of Theorem 1

Now that we have set up the players of our story, we will complete the proof of Theorem 1 as a series of lemmata.

Lemma 6. *The derived category of $\mathcal{O}'_{\mathfrak{p}_r}(\chi_0, \mathcal{W}^r_e, \mathfrak{q})$ embeds fully faithfully in the derived category of all $\mathcal{W}^r_e$-modules.*

Proof. This is essentially the same as the proof of [BGS96, 3.3.2]. It is enough to show that that $\mathrm{Ext}^{>0}_{\mathcal{W}^r_e}(P, I) = 0$ for P projective and I injective in $\mathcal{O}'_{\mathfrak{p}_r}(\eta, \mathcal{W}^r_e, \mathfrak{q})$. By the equivalence with category $\mathcal{O}$, the module P possesses a filtration by standard modules over the W-algebra, and I a filtration by their duals. These modules are the image under the Skryabin equivalence [Pre02] of the modules $M(\lambda, f)$ defined in [MS97, §2] and their duals. The higher Ext's between these standard and costandard modules vanish by Shapiro's Lemma, so we are done. □

Lemma 7. *On each component of $\Lambda_{V;\underline{\lambda}}$, there are no G_v-invariant functions of positive weight for the action of $\mathbb{T}$, and the only $\mathbb{T} \times G_v$-invariant functions are the constants.*

Proof. For any $(f_*, F^*_*) \in Z_{\mathbf{i},\kappa}$, there is a cocharacter into $G_v \times \mathbb{T}$ which attracts this point to the fiber over 0. We choose a list of rational numbers $g_1 < \cdots < g_m$ such that if $\kappa(k-1) < j \leq \kappa(k)$, we have $k - 1 < g_j < k$. Then, we multiply these by an integer q sufficiently large to clear the denominators. Pick an ordered homogeneous basis $(v_1, \ldots, v_m)$ of V which splits the flag F_*, and let $\varphi\colon \mathbb{C}^* \to G_v \times \mathbb{T}$ be defined by $\varphi(t) = (\mathrm{diag}(t^{qg_1}, t^{qg_2}, \ldots, t^{qg_m}), t^q)$.

If f_* is a quiver representation, then the matrix coefficient $v^*_i(f_e v_j)$ is 0 unless $i < j$; in this case, $\varphi(t)$ acts on this coefficent by $t^{qg_i - qg_j}$, which goes to 0 as $t \to \infty$ since $g_i < g_j$; similarly, for a new edge e_j, the coefficient $w^*(f_{e_j} v_i)$ can be non-zero only if $i > \kappa(j)$, in which case, the action is t^{qj-qg_i}, which likewise goes to 0 as $t \to \infty$.

Thus point in the image of $Z_{\mathbf{i}}$ which is compatible with the flag F_* has limit 0 as $t \to \infty$. Thus, any G_v-invariant function of positive weight for $\mathbb{T}$ must vanish at this point, and any $\mathbb{T} \times G_v$-invariant function must have the same value here as it does at 0. Since each point in the image is compatible with some flag, there can be no G_v-invariant functions on this image of positive weight for $\mathbb{T}$, and the only $\mathbb{T} \times G_v$-invariant functions are constant.

Since every component of $\Lambda_{V;\underline{\lambda}}$ is in the image of some $Z_{\mathbf{i}}$, there can be no non-zero G_v-invariant function of positive weight for $\mathbb{T}$ on any component, and the only invariant functions of zero weight are the constants. □

Lemma 8. *The functor h_η sends $Y_{\mathbf{i},\kappa}$ to the derived category $D^b(Q)$.*

Proof. In order to establish this, we must show that the Lie subalgebra $\mathfrak{t}$ acts locally finitely on the cohomology of $h(Y_{\mathbf{i},\kappa})$, with generalized weight spaces finite dimensional and weights bounded above.

The complex of D-modules $Y_{\mathbf{i},\kappa}$ is regular since it is of geometric origin. Since $\mathfrak{t}$ is a first order differential operator and its symbol vanishes on $\Lambda_{V;\underline{\lambda}}$, it preserves any very good filtration on $Y_{\mathbf{i},\kappa}$. Since each step of the filtration is finite-dimensional and the principal symbol of $\mathfrak{t}$ vanishes on $\Lambda_{V;\underline{\lambda}}$,the subtorus $\mathfrak{t}$ acts locally finitely on $Y_{\mathbf{i},\kappa}$.

Furthermore, its action on $Y_{\mathbf{i},\kappa}$ descends to the associated graded $\operatorname{gr} Y_{\mathbf{i},\kappa}$ for any very good filtration, as an infinitesimal action of $\mathbb{t}$, compatible with the $\mathbb{T}$-equivariant structure on the structure sheaf of T^*E_ν.

Obviously, we can replace $\operatorname{gr} Y_{\mathbf{i},\kappa}$ by its associated graded for any filtration without changing the properties we desire, so we may assume it is a quotient of the sum of finitely many copies of structure sheaves of components of $\Lambda_{V;\underline{\lambda}}$, with their $\mathbb{T}$-structure only twisted by a character.

By Lemma 7, the G_ν-invariant functions have $\mathbb{T}$-weights which are bounded above. The weight spaces must be finite dimensional since the invariant functions are finitely generated as an algebra. This completes the proof. □

Lemma 9. *Every simple in $\mathcal{O}'_{\mathfrak{p}_r}(\chi_0, \mathcal{W}^r_e, \mathfrak{q})$ is a summand of $h(Y_{\mathbf{i},\kappa})$ for some $\mathbf{i}$, and the functor h induces an isomorphism*

$$T^{\underline{\lambda}}_{\lambda-\nu} \cong \operatorname{Ext}_{A^\eta_\nu}\Big(\bigoplus_{\mathbf{i},\kappa} h_\eta(Y_{\mathbf{i},\kappa}), \bigoplus_{\mathbf{i}} h_\eta(Y_{\mathbf{i},\kappa})\Big).$$

Proof. First we note that localization holds (in the sense of [BPW, §4.2]) at the character χ_0 on the parabolic Springer resolution $T^*G/P \times_{\mathfrak{g}^*} \mathcal{S}_e$ for the Slodowy slice $\mathcal{S}_e$; by [Gin09, 5.2.4], the $\mathbb{Z}$-algebra arising from this choice of quantization is Morita for any ample line bundle on the resolution, and by [BPW, 5.9], this is equivalent to localization holding. Thus, the simple summands of $Y_{\mathbf{i},\kappa}$ which are killed by h_η are exactly those with unstable singular support. By [Li, 8.2.1(4)], those are exactly the summands of $Y_{\mathbf{i},\kappa}$ where $\kappa(1) > 0$. This shows that there are $\dim(V_{\underline{\lambda}})_{\lambda-\nu}$ simples in the essential image of h_η. This is the number of simples in the corresponding category $\mathcal{O}$, since it is also the number of fixed points of $\mathbb{T}$ acting on the corresponding quiver variety, so every simple is in the essential image.

Furthermore, this functor is full on simple objects. Let $\mathcal{Q}'$ denote the subcategory of G_ν-equivariant D_ν-modules such that all composition factors appear as summands of $(D_\nu/D_\nu \cdot (\mu + \chi)(\mathfrak{g}_\nu)) \otimes_{A^\chi_\nu} M$ for $M \in \mathcal{Q}$. By adding a finite number of simples to $\oplus Y_{\mathbf{i},\kappa}$, we obtain a semi-simple generator Y' of this category. The image of $\mathcal{Q}'$ in $\mathcal{Q}$ is the quotient category by the simples that are killed by h, that is, the category of dg-modules over the quotient of $\operatorname{Ext}(Y, Y)$ by the ideal generated by projections to simples that are killed by the quotient functor. Thus, the Ext-algebra of $\bigoplus_{\mathbf{i},\kappa} h_\eta(Y_{\mathbf{i},\kappa})$ is a quotient of $\operatorname{Ext}(Y_{\mathbf{i},\kappa}, Y_{\mathbf{i},\kappa}) \cong \tilde{T}^{\underline{\lambda}}$ which kills the violating ideal (but *a priori* have larger kernel). This shows we have a surjective map

$$q \colon T^{\underline{\lambda}} \to \operatorname{Ext}_{\mathcal{W}^r_e}\Big(\bigoplus_{\mathbf{i},\kappa} h_\eta(Y_{\mathbf{i},\kappa}), \bigoplus_{\mathbf{i},\kappa} h_\eta(Y_{\mathbf{i},\kappa})\Big).$$

On the other hand, we know by [Weba, 4.7] that there is *some* equivalence between $D(T^{\underline{\lambda}}\text{-dg-mod})$ and the Koszul dual of $\mathcal{Q}$. Thus, the dimension of the Ext algebra of an irredundant sum of simples coincides between the two categories. Since we already know that q is surjective between these, it must be an isomorphism, and we are done. □

Proof of Theorem 1. The category $\mathcal{Q}$ is Koszul. Thus, the Ext-algebra of any semi-simple generator is Koszul, and by Lemma 6, we can do this calculation in the whole

derived category of A^η_ν-modules. By Lemma 9, the tensor product algebra $T^{\underline{\lambda}}$ appears as such an algebra, and thus is Koszul. □

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